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ON THE EVOLUTION OF WEAK DISCONTINUITIES IN A STATE CHARACTERIZED BY INVARIANT SOLUTIONS

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Abstract—A quasilinear hyperbolic system which is invariant under scaling is transformed into a system which has a constant state in appropriate similar variables. These constant state solutions become special non-constant state solutions in the original variables. Two physical examples from gas dynamics and elastic–plastic deformations are studied and the occurrence of shock waves demonstrated.

1. INTRODUCTION

First order quasilinear hyperbolic systems

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial}{\partial x} F(\mathbf{U}) = \mathbf{B}(\mathbf{U}, x, t) \tag{1.1}$$

arise in wave propagation problems in cylindrical or spherical symmetry as well as in regions with inhomogeneities. In a previous paper [1] it was assumed that (1.1) was invariant with respect to an infinitesimal group of transformations. For a similar (invariant) line to be a discontinuity line for the first derivatives of U, such that the jump conditions across the lines involves similar variables only, certain compatibility conditions are required. In the present paper we consider the problem of the propagation of weak discontinuities, compatible with (1.1), in a non-constant state characterized by a known similar solution. Studies of blast waves in a fluid give rise to problems of this type. The self-similar solution of the point explosion problem is only valid for a short time. Then a secondary shock may appear [2]. The evolution of this special self-similar flow, characterized by Sedov [3], has been studied by several authors [2, 4, 5].

In the first part of this paper the governing system of equations is assumed to be invariant with respect to the dilatation group. Under a suitable change of variables the resulting evolution equation of weak discontinuities is integrated in terms of the similar variables. The constant state solutions of the transformed system become special non-constant solutions in the original variables. The Sedov solution, mentioned above, is a member of this class.

Using this procedure exact solutions for the problem of elastic–plastic deformations [6] generated by a torque suddenly applied to a rigid cylinder embedded in an infinite elastic-plastic medium are obtained. The torque produces plastic deformations that propagate into the surrounding medium. The propagation of weak discontinuities is studied in this special non-constant state and the occurrence of shock waves is demonstrated.

2. INVARIANCE CONDITIONS AND THE "REDUCED" SYSTEM

Many systems (1.1) of physical interest (cf. [7] or [8]) are invariant under the dilatation group

$$t^* = \mu^{\gamma} t, \qquad x^* = \mu x, \qquad u_A^* = \mu^{\alpha_A} u_A, \qquad A = 1, 2, \ldots, N. \tag{2.1}$$

In what follows it is assumed that at least one of the α_A is different from zero, say $\alpha_I = \alpha$, with the corresponding component of $\mathbf{U}$ denoted by $u_I = u$. The invariance of (1.1) with respect to (2.1) requires the following expressions for F and B

$$
\begin{aligned}
F_L &= u^{\rho_L/\alpha}\tilde{f}_L(U_1, U_2, \ldots, 1, \ldots, U_N) \\
B_L &= u^{(\rho_L - 1)/\alpha}\tilde{b}_L(U_1, U_2, \ldots, 1, \ldots, U_N, x/u^{1/\alpha}, t/u^{\gamma/\alpha})
\end{aligned}
\tag{2.2}
$$

where

$$
U_L = u_L/u^{\alpha_L/\alpha}, \qquad \rho_L = \alpha_L - \gamma + 1, \qquad L = 1, 2, \ldots, N.
$$

To this system the following transformation of variables is applied

$$
\tau = \ln t, \qquad \xi = x/t^{1/\gamma}, \qquad u_L = t^{\alpha_L/\gamma}V_L(\xi, \tau)
\tag{2.3}
$$

such that

$$
U_L = V_L/V^{\alpha_L/\alpha}, \qquad V_I = V, \qquad u = t^{\alpha/\gamma}V,
$$

whereupon (2.2) becomes

$$
\begin{aligned}
F_L &= t^{\rho_L/\gamma}V^{\rho_L/\alpha}\tilde{f}_L = t^{\rho_L/\gamma}\tilde{F}_L(U_1, U_2, \ldots, 1, \ldots, U_N) \\
B_L &= t^{(\rho_L - 1)/\gamma}V^{(\rho_L - 1)/\alpha}\tilde{b}_L \\
&= t^{(\rho_L - 1)/\gamma}\tilde{B}_L(U_1, \ldots, 1, \ldots, U_N, \xi/V^{1/\alpha}, V^{-\gamma/\alpha}).
\end{aligned}
\tag{2.4}
$$

In the new variables, (2.3), the governing system (1.1) in component form, becomes

$$
\frac{\partial V_L}{\partial \tau} - \frac{\xi}{\gamma}\frac{\partial V_L}{\partial \xi} + \frac{\partial \tilde{F}_L}{\partial \xi} = \tilde{B}_L - \frac{\alpha_L}{\gamma}V_L.
\tag{2.5}
$$

Moreover, assuming that $\tilde{B}_L$ may be written as

$$
\tilde{B}_L = \frac{\hat{B}_L}{\xi} = \frac{V^{\rho_L/\alpha}}{\xi}\tilde{b}_L\left(U_1, \ldots, 1, \ldots, U_N, \frac{V^{(1-\gamma)/\alpha}}{\xi}\right)
\tag{2.6}
$$

—i.e. a combination of the last two terms in (2.4), the equation can model physical examples of interest with cylindrical or spherical symmetry. In the new variables

$$
V_L = \xi^{\alpha_L/(1-\gamma)}W_L, \qquad \eta = \ln \xi
\tag{2.7}
$$

the equation (2.5) becomes

$$
\frac{\partial W_L}{\partial \tau} + \frac{\partial \hat{F}_L}{\partial \eta} - \frac{1}{\gamma}\frac{\partial W_L}{\partial \eta} = \hat{B}_L + \frac{\alpha_L}{1-\gamma}W_L - \frac{\rho_L}{1-\gamma}\hat{F}_L,
\tag{2.8}
$$

where $\hat{F}_L = \tilde{F}_L(W_L)$, $\hat{B}_L = \tilde{B}_L(W_L)$.

Consequently, under the assumed conditions, the transformations (2.3) and (2.7) change (1.1) into (2.8), with independent variables τ and η which do not explicitly appear in the coefficients. It should be noted here that condition (2.6) essentially means that system (2.5) is invariant with respect to the dilatation group $\tau^* = \tau$, $\xi^* = \omega\xi$, $V_L^* = \omega^{\alpha_L/(1-\gamma)}V_L$, $\omega \in \mathbb{R} - \{0\}$.

3. EVOLUTION OF DISCONTINUITIES

Here is studied the propagation of weak discontinuities, in the field variables W_L compatible with the system (2.8), across a curve $\phi(\tau,\eta)=0$ in a non-constant state characterized by a similarity solution $W_{0L}(\eta)$ satisfying

$$-\frac{1}{\gamma}\frac{\mathrm{d}W_{0L}}{\mathrm{d}\eta}+\frac{\mathrm{d}\hat{F}_{0L}}{\mathrm{d}\eta}=\hat{B}_{0L}+\frac{\alpha_L}{1-\gamma}W_{0L}-\frac{\rho_L}{1-\gamma}\hat{F}_{0L}, \tag{3.1}$$

where "0" means that a quantity is evaluated for $W_L=W_{0L}$.

This problem requires that the non-constant state, in which the discontinuity propagates, be dependent only on one independent variable called the "space" variable. For simplicity the system (2.8) studied hereafter will be strictly hyperbolic so that the characteristic velocities, which are solutions† of

$$\mathrm{Det}\left[\mathrm{A}_{LM}-\left(\hat{\lambda}+\frac{1}{\gamma}\right)\delta_{LM}\right]=0 \tag{3.2}$$

$$\hat{\lambda}=-\frac{\phi_\tau}{\phi_\eta}=\frac{\mathrm{d}\eta}{\mathrm{d}\tau},\qquad A_{LM}=\frac{\partial\hat{F}_L}{\partial W_M}, \tag{3.3}$$

are distinct real numbers and the corresponding right, $\mathbf{d}_M$, and left, $\mathbf{l}_L$, eigenvectors of A_{LM} are linearly independent.

Under these assumptions and with $d_{0L}=d_L(W_{0L})$,

$$\delta=\left.\frac{\partial}{\partial\phi}\right|_{\phi=0^+}-\left.\frac{\partial}{\partial\phi}\right|_{\phi=0^-}$$
$$\delta W_L=\pi d_{0L}, \tag{3.4}$$

where π satifies the well known evolution equations [9],

$$(\mathbf{l}_0\cdot\mathbf{d}_0)\frac{\mathrm{d}\pi}{\mathrm{d}\sigma}+(\nabla_{\mathbf{W}}\hat{\lambda}\cdot\mathbf{d})_0\phi_\eta(\mathbf{l}_0\cdot\mathbf{d}_0)\pi^2$$
$$+\left\{\left(\mathbf{l}_0\frac{\mathrm{d}\mathbf{W}_0}{\mathrm{d}\eta}\right)(\nabla_{\mathbf{W}}\hat{\lambda}\cdot\mathbf{d})_0+\left(\mathbf{l}\cdot\nabla_{\mathbf{W}}\mathbf{d}\frac{\mathrm{d}\mathbf{W}_0}{\mathrm{d}\eta}\right)_0+\hat{\lambda}_0(\nabla_{\mathbf{W}}\mathbf{l}\cdot\mathbf{d})_0\frac{\mathrm{d}\mathbf{W}_0}{\mathrm{d}\eta}\right\}\pi$$
$$=\nabla_{\mathbf{W}}(\mathbf{l}\cdot\mathcal{B})_0\mathbf{d}_0\pi, \tag{3.5}$$

where

$$\mathcal{B}_L=\hat{B}_L+\frac{\alpha_L}{1-\gamma}W_L-\frac{\rho_L}{1-\gamma}\hat{F}_L=\mathcal{B}_L(W_M)$$
$$\frac{\mathrm{d}}{\mathrm{d}\sigma}=\frac{\partial}{\partial\tau}+\hat{\lambda}_0\frac{\partial}{\partial\eta}. \tag{3.6}$$

The characteristics associated with (2.8) are

$$\tau=\sigma,\qquad \tau-\tau_0=\int_{\eta_0}^{\eta}\frac{\mathrm{d}\eta}{\hat{\lambda}_0(\eta)}, \tag{3.7}$$

† δ_{LM} is the Kronecker symbol.

so that

$$\phi = \phi\left(\tau - \int_{\eta_0}^{\eta} \frac{\mathrm{d}\eta}{\hat{\lambda}_0(\eta)}\right) \tag{3.8}$$

may be determined from its critical conditions.

Since equation (3.5) is of Bernoulli form,

$$\frac{\mathrm{d}\pi}{\mathrm{d}\sigma} + a(\eta)\pi^2 = b(\eta)\pi,$$

it is easily integrated to give

$$\pi = h/\phi, \tag{3.9}$$

where

$$h = \pi_0 \exp \int_{\eta_0}^{\eta} b(\eta)\mathrm{d}\eta/\hat{\lambda}_0(\eta)$$

$$\phi = \int_{\eta_0}^{\eta} a(\eta)h\,\mathrm{d}\eta/\hat{\lambda}_0(\eta) + 1. \tag{3.10}$$

In the original variables u_L is given by

$$u_L = \left(\frac{x}{t}\right)^{\alpha_L/(1-\gamma)} W_L = t^{\alpha_L/\gamma} V_L \tag{3.11}$$

and its first derivatives are discontinuous across $\phi^*(x, t) = \phi[\tau(t), \eta(x, t)]$. With

$$\delta^* = \left.\frac{\partial}{\partial \phi^*}\right|_{\phi^* = 0^+} - \left.\frac{\partial}{\partial \phi^*}\right|_{\phi^* = 0^-}$$

there follows

$$\delta^* U_L = \left(\frac{x}{t}\right)^{\alpha_L/(1-\gamma)} \delta W_L = \left(\frac{x}{t}\right)^{\alpha_L/(1-\gamma)} d^*_{0L}\pi. \tag{3.12}$$

Moreover as

$$\frac{\partial}{\partial t} = \frac{1}{t}\frac{\partial}{\partial \tau} - \frac{1}{\gamma t}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial x} = \frac{1}{x}\frac{\partial}{\partial \eta}$$

there follows that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda = \frac{x}{t}\left(\hat{\lambda} + \frac{1}{\gamma}\right). \tag{3.13}$$

All the information is now available to calculate the discontinuities of the field variables in the cases considered.

4. PROPAGATION IN "CONSTANT STATE"—PHYSICAL APPLICATIONS

Let $W_L = W_{0L} =$ constant be a solution of

$$\mathscr{B}_L(W_{0L}) = \hat{B}_{0L} + \frac{\alpha_L}{1-\gamma} W_{0L} - \frac{\rho_L}{1-\gamma} \hat{F}_{0L}.$$

Because of (3.11) this corresponds to a non-constant particular solution of the original system (1.1)—in fact a self similar solution. As will be seen, the propagation of weak discontinuities in the original variables in such "non-constant" states may be studied as a propagation in a "constant state" in the new variables. In the present special case $\tau = \sigma$, $\eta = \eta_0 + \hat{\lambda}_0(\tau - \tau_0)$, $\hat{\lambda}_0 =$ constant and $\sigma = \ln t$,

$$x/x_0 = (t/t_0)^{\hat{\lambda}_0 + \gamma^{-1}}.$$

Consequently, from (3.9) and (3.10), there results

$$\pi = \frac{\pi_0 (t/t_0)^{b_0}}{1 + \pi_0 \dfrac{a_0}{b_0}\{(t/t_0)^{b_0} - 1\}}. \tag{4.1}$$

From (4.1) the time t_c when the discontinuity wave evolves into a shock wave is given by

$$t_c = t_0 \left[\frac{a_0 \pi_0 - b_0}{\pi_0 a_0}\right]^{1/b_0}. \tag{4.2}$$

This occurs when the initial discontinuity is such that

(i) $a_0\pi_0 < 0$, $a_0\pi_0 < b_0$; (ii) $a_0\pi_0 > 0$, $a_0\pi_0 > b_0$.

A. As a *first example* a spherical symmetric motion of a polytropic gas neglecting any dissipation mechanism is studied. The equations of motion take the form of (1.1) with

$$\mathbf{U} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, \quad A = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \Gamma p & u \end{bmatrix}, \quad B = \begin{bmatrix} -2u\rho/r \\ 0 \\ -2\Gamma u p/r \end{bmatrix}, \tag{4.3}$$

where ρ is the density, u is the particle velocity, p the gas pressure, r the distance from the center of symmetry and Γ is the gas index.

Upon identifying $u_1 = \rho$, $u_2 = u$, $u_3 = p$ there is invariance with respect to the dilatation group (2.1) provided that

$$\alpha_2 = 1 - \gamma, \qquad \alpha_3 = \alpha_1 + 2(1-\gamma). \tag{4.4}$$

If the initial density of mass in the medium at rest is

$$\rho_0(r) = \bar{\rho}_0 r^{-\Omega}, \qquad \Omega = \frac{7-\Gamma}{1+\Gamma},$$

then $\alpha_1 = \alpha = -\Omega$. Moreover if a strong explosion takes place at the origin then, considering that motion to be self similar, it follows from energy conservation that γ is given by $\gamma = (3\Gamma - 1)/(\Gamma + 1)$.

Under the transformations

$$\begin{gathered} p(r,t) = \bar{\rho}_0 t^{\alpha/\gamma} R(t) \mathscr{P}(\xi,\tau) \\ \rho(r,t) = \bar{\rho}_0 t^{\alpha/\gamma} g(\xi,\tau) \\ u(r,t) = R v(\xi,\tau) \\ \xi = r/R(t), \qquad R(t) = t^{1/\gamma}, \qquad \tau = \ln t, \qquad z = \Gamma \mathscr{P}/g, \end{gathered} \tag{4.5}$$

the system (4.3) becomes

$$\frac{\partial \mathbf{V}}{\partial \tau} + A \frac{\partial \mathbf{V}}{\partial \xi} = \mathscr{B}(\mathbf{V}, \xi), \tag{4.6}$$

where

$$\mathbf{V} = \begin{bmatrix} g \\ v \\ z \end{bmatrix}, \qquad A = \begin{bmatrix} (v-\xi)/\gamma, & g/\gamma, & 0 \\ \dfrac{z}{\gamma g \Gamma} & (v-\xi)/\gamma, & 1/\gamma\Gamma \\ 0 & (\Gamma-1)z/\gamma, & (v-\xi)/\gamma \end{bmatrix}$$

and

$$\mathscr{B} = \begin{bmatrix} -g\left(2v/\xi + \dfrac{\Gamma-7}{\Gamma+1}\right)/\gamma \\ 2v\left(\dfrac{\Gamma-1}{\Gamma+1}\right)/\gamma \\ -2zv(\Gamma-1)/\gamma\xi + 4z(\Gamma-1)/\gamma(\Gamma+1) \end{bmatrix}.$$

In fact this is the case studied in [2, 4, 5].

System (4.6) is invariant with respect to the group

$$\begin{gathered} g^* = \omega^A g, \qquad v^* = \omega v, \qquad z^* = \omega^2 z \\ \tau^* = \tau, \qquad \xi^* = \omega \xi. \end{gathered} \tag{4.7}$$

The new variables transformation

$$g = \xi G, \qquad v = \xi V, \qquad z = \xi^2 Z, \qquad \eta = \ln \xi, \tag{4.8}$$

changes (4.6) into the form

$$\frac{\partial \mathbf{W}}{\partial \tau} + \hat{A} \frac{\partial}{\partial \eta} \mathbf{W} = \hat{\mathscr{B}}, \tag{4.9}$$

where

$$\mathbf{W} = \begin{bmatrix} G \\ V \\ Z \end{bmatrix}, \qquad \hat{A} = \begin{bmatrix} (V-1)/\gamma, & G/\gamma, & 0 \\ Z/\gamma\Gamma G, & (V-1)/\gamma, & 1/\gamma\Gamma \\ 0 & (\Gamma-1)Z/\gamma, & (V-1)/\gamma \end{bmatrix}$$

$$\hat{\mathscr{B}} = \begin{bmatrix} 4G\{2/(\Gamma+1) - V\}/\gamma \\ 2V(\Gamma-1)/\gamma(\Gamma+1) - V(V-1)/\gamma - 3Z/\gamma\Gamma \\ Z(3\Gamma-1)\{(2/(\Gamma+1) - V\}/\gamma \end{bmatrix}.$$

System (4.9), also taking into account the Rankine–Hugoniot relations for strong shocks, has a constant state solution given by

$$G = G_0 = \frac{\Gamma + 1}{\Gamma - 1}, \qquad V = V_0 = \frac{2}{\Gamma + 1}, \qquad Z = Z_0 = \frac{\cancel{2}\Gamma(\Gamma - 1)}{(\Gamma + 1)^2}. \tag{4.10}$$

This solution, when substituted into (4.8), gives rise to the Sedov solution [3].

Routine calculations generate the eigenvalues of $\hat{A}$

$$\lambda_1 = (V - 1 + Z^{1/2})/\gamma, \qquad \lambda_2 = (V - 1)/\gamma, \qquad \lambda_3 = (V - 1 - Z^{1/2})/\gamma.$$

The values a_0, b_0 of (4.1) and (4.2) corresponding to λ_1 and λ_3 are

$$a_0 = \pm\frac{1}{8}\frac{(\Gamma + 1)^2}{4\Gamma^2(\Gamma - 1)}[2\Gamma(\Gamma - 1)]^{1/2}$$

$$b_0 = \frac{1}{4(3\Gamma - 1)\Gamma^2}\left\{\pm\frac{[2\Gamma(\Gamma - 1)]^{1/2}}{\Gamma - 1}(\Gamma + 1)(1 - 3\Gamma)(1 + 3\Gamma) + 2\Gamma^2(3\Gamma - 5)\right\}.$$

Equation (4.1) is valid for all cases except that one for which $b_0 = 0$. Denoting by Γ_0 that value of Γ for which $b_0 = 0$ the corresponding solution for π is

$$\pi = \pi_0/(1 + \pi_0 \ln(t/t_0)^{a_0(\Gamma_0)}). \tag{4.11}$$

B. As a *second application* consider a rigid cylinder of radius $r = r_0$ embedded in an infinite elastic-plastic medium with the entire system initially undeformed and at rest. At a certain moment a sudden torque is applied to the cylinder which produces deformations which propagate into the surrounding medium. The only equation of motion [6] is

$$\rho W_{tt} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Sigma) = 0, \tag{4.12}$$

where W is the circumferential displacement, $\mathbf{\Sigma}$ the stress and ρ is the density. Assuming a constitutive relation of the type $\Sigma = \mathbf{\Sigma}/\rho$, $\Sigma = \Sigma_1\varepsilon^p$, $\varepsilon = \frac{W}{r} - \frac{\partial W}{\partial r}$, and choosing as new field variables $v = W_t$ and ε, equation (4.12) becomes the first order system

$$v_t + \Sigma_r = -2\Sigma/r, \qquad \varepsilon_t + v_r = v/r. \tag{4.13}$$

This system has the exact solutions

$$v = E_0\left(\frac{r}{t}\right)^{(p+1)/(p-1)}, \qquad \varepsilon = E_0\left(\frac{r}{t}\right)^{2/(p-1)}, \tag{4.14}$$

where

$$E_0 = \left[\frac{p + 1}{2(2p - 1)}\frac{1}{\Sigma_1}\right]^{1/(p-1)}$$

and p has the restrictions

$$\frac{1}{2} < p < 1 \quad \text{or} \quad -1 < p < 0$$

in order for the system to be hyperbolic. In addition the following initial and boundary conditions must be satisfied [6]:

$$t > 0, \quad r_0 \le r < \infty, \quad \varepsilon = \varepsilon_0 t^{-2/(p-1)}$$

$$t = 0, \quad r_0 \le r < \infty, \quad w = 0, \quad w_r = 0.$$

The displacement W is given by

$$W = -\frac{1}{2}(p-1)E_0 r^{(p+1)/(p-1)} t^{-2/(p-1)}.$$

The study of the propagation of weak discontinuities in a state characterized by the solution (4.14) is given in the sequel. The speed of propagation of the discontinuity is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \lambda^{\pm} = \pm\frac{r}{t}\left[\frac{p(p+1)}{p(2p-1)}\right]^{1/2}. \tag{4.15}$$

From calculations similar to those of Part A the values for a_0 and b_0 are found to be

$$a_0 = \pm\left[\frac{p(p+1)}{2(2p-1)}\right]^{1/2}\frac{p-1}{2E_0}$$

$$b_0 = \frac{1}{p-1}\left\{\frac{1}{2}(p+3) - 2p\left(\pm\left[\frac{p(p+1)}{2(2p-1)}\right]^{1/2}\right)\right\}. \tag{4.16}$$

Three cases will be distinguished.

(i) $\lambda = \lambda^+$, $-1 < p < 0$.

Here $a_0 < 0$, $b_0 < 0$ and shocks will appear either if $\pi_0 > 0$ when $\pi_0 > |b_0|/|a_0|$ or if $\pi_0 < 0$.

(ii) $\lambda = \lambda^+$, $\frac{1}{2} < p < 1$.

Then $a_0 < 0$ and $b_0 > 0$. So shocks will appear if $\pi_0 > 0$ or when $|\pi_0| > b_0/|a_0|$ if $\pi_0 < 0$.

(iii) $\lambda = \lambda^-$.

Then $a_0 > 0$ and $b_0 < 0$ and there is the possibility of shocks if $\pi_0 > 0$ or when $|\pi_0| > |b_0|/a_0$ if $\pi_0 < 0$.

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